

A rigidity theorem for quaternionic Kähler structures

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Abstract

We study the moduli space of quaternionic Kähler structures on a compact manifold of dimension $4n \geq 12$ from a point of view of Riemannian geometry, not twistor theory. Then we obtain a rigidity theorem for quaternionic Kähler structures of nonzero scalar curvature by observing the moduli space.

1 Introduction

According to Berger's classification theorem, the holonomy group of a simply-connected, non-symmetric, irreducible Riemannian manifold of dimension N is isomorphic to one of the following;

$$SO(N), U(N/2), SU(N/2), Sp(N/4), Sp(N/4)Sp(1), G_2, Spin(7).$$

The Riemannian geometry of special holonomy groups $SU(N/2)$, $Sp(N/4)$, $Sp(N/4)Sp(1)$, G_2 and $Spin(7)$ are called Calabi-Yau, hyperKähler, quaternionic Kähler, G_2 and $Spin(7)$ structures, respectively. During the last score of twentieth century, the deformation theory of these structures are studied according to individual way to each structure. For example, the deformations of Calabi-Yau and hyperKähler structures were studied by using Kodaira-Spencer theory [1][20][21]. But we cannot apply Kodaira-Spencer theory to the other structures since they do not admit complex structures. Joyce showed that the moduli spaces of G_2 and $Spin(7)$ structures are smooth manifolds by studying the closed differential forms which define the structures. Then the purpose of this paper is studying the moduli spaces of the quaternionic Kähler structures.

Each quaternionic Kähler structure has an Einstein metric. If the metric is Ricci-flat, then it reduces to a hyperKähler structure. So, if we denote by κ_g the scalar curvature of a Riemannian metric g , we should consider the case of $\kappa_g > 0$ or $\kappa_g < 0$ for studying quaternionic Kähler structures.

A Riemannian metric g on $4n$ -dimensional manifold M is a quaternionic Kähler metric if the holonomy group of g is isomorphic to a subgroup of $Sp(n)Sp(1)$. Then there are rigidity theorems for the quaternionic Kähler metrics as follows.

Theorem 1.1 ([10]). *Let M be a compact $4n$ -manifold, $n \geq 2$, and let $\{g_t\}$ be a family of quaternionic Kähler metrics on M , of fixed volume, depending smoothly on $t \in \mathbb{R}$. If g_0 has positive scalar curvature then there is a family of diffeomorphisms $\{\psi_t\} \subset \text{Diff}(M)$ depending smoothly on $t \in \mathbb{R}$ such that $\psi_t^* g_t = g_0$.*

Moreover, LeBrun and Salamon [11] showed that there are, up to isometries and rescalings, only finitely many compact quaternionic Kähler metrics of dimension $4n$ of positive scalar curvature for each $n \geq 2$.

Theorem 1.2 ([7]). *Let M be a compact $4n$ -manifold and let $\{g_t\}$ be a family of quaternionic Kähler metrics on M , of fixed volume, depending smoothly on $t \in \mathbb{R}$. If g_0 has negative scalar curvature then there is a family of diffeomorphisms $\{\psi_t\} \subset \text{Diff}(M)$ depending smoothly on $t \in \mathbb{R}$ such that $\psi_t^* g_t = g_0$.*

The above two theorems are proven by using twistor theory.

In this paper, we will prove the rigidity for quaternionic Kähler structures in the case of $\kappa_g > 0$ and $\kappa_g < 0$ at the same time using Riemannian geometry without using twistor theory.

We apply [5] to the description of the moduli spaces of quaternionic Kähler structures. In [5], Goto introduced a notion of topological calibration which gives a unified framework of the deformation theory of Calabi-Yau, hyperKähler, G_2 and $Spin(7)$ structures. The moduli space of topological calibrations is constructed in Riemannian geometric way.

We define the set of quaternionic Kähler structures of nonzero scalar curvature on M in Section 3 and denote it by $\widetilde{\mathcal{M}}_{qK}$. Since $\widetilde{\mathcal{M}}_{qK}$ is a subset of closed 4-forms on M , then $\mathcal{G} := \text{Diff}_0(M) \times \mathbb{R}_{>0}$ acts on $\widetilde{\mathcal{M}}_{qK}$ by the pull-back and scalar multiple. So we have a quotient space $\mathcal{M}_{qK} := \widetilde{\mathcal{M}}_{qK}/\mathcal{G}$ and the quotient map $\pi_{qK} : \widetilde{\mathcal{M}}_{qK} \rightarrow \mathcal{M}_{qK}$. We will show a rigidity theorem for quaternionic Kähler structures as follows.

Theorem 1.3. *Let $\{\Phi_t\}_{t \in \mathbb{R}} \in \widetilde{\mathcal{M}}_{qK}$ be a continuous family of quaternionic Kähler structures on compact $4n$ -dimensional manifold M for $n \geq 3$. Then we have $\pi_{qK}(\Phi_t) = \pi_{qK}(\Phi_0)$ for any $t \in \mathbb{R}$.*

To show Theorem 1.3, we have to evaluate the dimension of the formal tangent space of \mathcal{M}_{qK} at $\pi_{qK}(\Phi)$. In Section 2, we introduce the deformation complex of quaternionic Kähler structures

$$\cdots \xrightarrow{d} \Gamma(E_\Phi^k) \xrightarrow{d} \Gamma(E_\Phi^{k+1}) \xrightarrow{d} \cdots$$

for each $\Phi \in \widetilde{\mathcal{M}}_{qK}$ along [5]. If we denote the k -th cohomology group of the above complex by $H^k(\sharp_\Phi)$, then the formal tangent space of \mathcal{M}_{qK} is given by $H^1(\sharp_\Phi)/\mathbb{R}$. To prove the rigidity theorem, we need to show that

- (I) the deformation complexes are elliptic complexes,
- (II) $H^1(\sharp_\Phi) \cong \mathbb{R}$.

It is shown that (I) is true in the case of Calabi-Yau, hyperKähler, G_2 and $Spin(7)$ structures in [5]. But if we try to show (I) in the case of quaternionic Kähler structures, we have to deal with 4-forms or 5-forms of $4n$ -dimensional vector space, which are so complicated. Hence we need more systematic method to study the deformation complex. Then we introduce a new method for showing (I) in Section 4.

In Section 4, we introduce new complexes called the deformation complexes of torsion-free $Sp(n)Sp(1)$ -structures, and show they are the elliptic complexes. Then we solve (I) by constructing the isomorphisms between the deformation complexes of quaternionic Kähler structures and new ones. Since these discussions can be applied to the other structures, we can regard the results in Section 4 as the unified method to study the deformation complexes of topological calibrations.

We prove (II) in Section 5 by using the Bochner-Weitzenböck formulas and vanishing theorems on quaternionic Kähler manifolds due to Homma [6], Semmelmann and Weingart [14].

Each quaternionic Kähler structure $\Phi \in \widetilde{\mathcal{M}}_{qK}$ induces a Riemannian metric g_Φ on M . If g is a quaternionic Kähler metric of nonzero scalar curvature, then there is a quaternionic Kähler structure $\Phi \in \widetilde{\mathcal{M}}_{qK}$ such that $g_\Phi = g$. Then, it is important to study how many quaternionic Kähler structures which induce a given quaternionic Kähler metric g . We will obtain the following theorem in Section 6.

Theorem 1.4. *Let (M, g) be a $4n$ -dimensional Riemannian manifold for $n \geq 3$ and $\widetilde{\mathcal{M}}_{qK}(g) := \{\Phi \in \widetilde{\mathcal{M}}_{qK}; g_\Phi = g\}$. If g is a quaternionic Kähler metric, then there is a unique element in $\widetilde{\mathcal{M}}_{qK}(g)$.*

2 Geometric structures defined by closed differential forms

In this section, we introduce Goto's topological calibration theory along [5], then state its relation to torsion-free G -structures.

Let g_0 be the standard inner product on $V = \mathbb{R}^N$. We have $GL_N\mathbb{R}$ representation $\rho : GL_N\mathbb{R} \rightarrow GL(\Lambda^k)$ by putting $\rho(g)\alpha := (g^{-1})^*\alpha$ for $g \in GL_N\mathbb{R}$ and $\alpha \in \Lambda^k := \Lambda^k V^*$. Fix

$$\Phi^V \in \bigoplus_{i=1}^l \Lambda^{p_i}$$

such that the isotropy group

$$G = \{g \in GL_N\mathbb{R}; \rho(g)\Phi^V = \Phi^V\}$$

is a subgroup of the orthogonal group $O(N)$.

In this section, we consider a smooth manifold M of dimension N . We denote by $\pi_{F(M)} : F(M) \rightarrow M$ the frame bundle of M whose fibre is $GL_N\mathbb{R}$. If we set

$$R_{\Phi^V}(V) := \{\rho(g)\Phi^V \in \bigoplus_{i=1}^l \Lambda^{p_i}; g \in GL_N\mathbb{R}\},$$

then there is a left action of $GL_N\mathbb{R}$ on $R_{\Phi^V}(V)$, given by $g_1 \cdot \rho(g_2)\Phi^V := \rho(g_1 g_2)\Phi^V$ for $g_1, g_2 \in GL_N\mathbb{R}$. Then we have an $R_{\Phi^V}(V)$ -bundle

$$R_{\Phi^V}(M) := F(M) \times_{GL_N\mathbb{R}} R_{\Phi^V}(V).$$

Since $R_{\Phi^V}(M)$ is a subbundle of $\bigoplus_{i=1}^l \Lambda^{p_i} T^*M$, we can consider the exterior derivative $d\Phi \in \bigoplus_{i=1}^l \Omega^{p_i+1}(M)$ for each $\Phi \in \Gamma(R_{\Phi^V}(M))$. Then we put

$$\widetilde{\mathcal{M}}_{\Phi^V}(M) = \{\Phi \in \Gamma(R_{\Phi^V}(M)); d\Phi = 0\}.$$

By taking proper N and Φ^V , we can construct the set of Calabi-Yau, hyperKähler, G_2 and $Spin(7)$ structures on M in this manner [5]. We will give $\Phi_{qK} \in \Lambda^4(\mathbb{R}^{4n})^*$ which determines the set of quaternionic Kähler structures in Section 3.

Next we see that there is one-to-one correspondence between torsion-free G -structures on M and $\widetilde{\mathcal{M}}_{\Phi^V}(M)$ under a certain condition for Φ^V .

Since G is a subgroup of $GL_N\mathbb{R}$, we have a quotient space

$$R_G(M) := F(M)/G,$$

which is a $GL_N\mathbb{R}/G$ -bundle over M . Then for each section $Q \in \Gamma(R_G(M))$, there is a principal G -bundle

$$\tilde{Q} := \{u \in F(M); \pi_G(u) = Q_{\pi_{F(M)}(u)}\},$$

where $\pi_G : F(M) \rightarrow F(M)/G$ is the quotient map.

By taking a section $Q \in \Gamma(R_G(M))$, we may write $TM = \tilde{Q} \times_G V$ where $V = \mathbb{R}^N$. Then a Riemannian metric g_Q on M is induced by $g_Q|_p(u \times_G x, u \times_G y) = g_0(x, y)$ for $x, y \in V$, $p \in M$ and $u \in \tilde{Q}_p$. Since G is a subgroup of $O(N)$, this is well-defined.

Definition 2.1. Let $Q \in \Gamma(R_G(M))$. A covariant derivative ∇ on TM is a connection on Q if ∇ is reducible to a connection on a principal G -bundle \tilde{Q} .

Definition 2.2. We call $Q \in \Gamma(R_G(M))$ a torsion-free G -structure if the Levi-Civita connection of g_Q is a connection on Q .

The natural diffeomorphism $R_{\Phi V}(V) \cong GL_N\mathbb{R}/G$ induces a bijective bundle map

$$\sigma_{\Phi V} : R_{\Phi V}(M) \longrightarrow R_G(M).$$

We put $Q_\Phi := \sigma_{\Phi V}(\Phi) \in \Gamma(R_G(M))$ for each $\Phi \in \Gamma(R_{\Phi V}(M))$.

We set a G -equivariant map $A_{\Phi V}^k : \Lambda^k \otimes V \rightarrow \bigoplus_{i=1}^l \Lambda^{p_i+k-1}$ as $A_{\Phi V}^k(\omega \otimes v) := \omega \wedge \iota_v \Phi^V$ for $\omega \otimes v \in \Lambda^k \otimes V$, where ι is the interior product, and put $E_{\Phi V}^k := \text{Im}(A_{\Phi V}^k)$. Then a bundle map $A_\Phi^k : \Lambda^k T^*M \otimes TM \rightarrow E_\Phi^k$ is induced by $A_{\Phi V}^k$ for each $\Phi \in \Gamma(R_{\Phi V}(M))$, where we put $E_\Phi^k := \tilde{Q}_\Phi \times_G E_{\Phi V}^k$.

Proposition 2.3. Let ∇ be a connection on Q_Φ for $\Phi = (\Phi_1, \dots, \Phi_l) \in \Gamma(R_{\Phi V}(M))$. Then we have

$$d\Phi = A_\Phi^2(T^\nabla),$$

where T^∇ is the torsion tensor of ∇ .

Proof. We calculate $(d\Phi)_p$ for a fixed point $p \in M$. Let ∇ be any connection on Q_Φ , $v_1, v_2, \dots, v_N \in V$ be an orthonormal basis and $v^1, v^2, \dots, v^N \in V^*$ be its dual basis.

We can take a neighborhood U of p and local section $\tau \in \Gamma(U, \tilde{Q}_\Phi)$ which satisfy $(\nabla \xi_i)_p = 0$, where $\xi_i|_x = \tau(x) \times_G v_i$ for $x \in U$.

Let $\Phi_l^V = \sum_{i_1, \dots, i_{p_l}} \Phi_{i_1, \dots, i_{p_l}}^{(l)} v^{i_1} \wedge \dots \wedge v^{i_{p_l}}$ ($l = 1, \dots, N$) and $\xi^i|_x = \tau(x) \times_G v^i$. Then for any $x \in U$, we have

$$\begin{aligned} (\Phi_l)_x &= \sigma(x) \times_G \Phi_l^V \\ &= \sum_{i_1, \dots, i_{p_l}} \Phi_{i_1, \dots, i_{p_l}}^{(l)} (\xi^{i_1})_x \wedge \dots \wedge (\xi^{i_{p_l}})_x. \end{aligned}$$

If we put $d\xi^\alpha = c_{\beta\gamma}^\alpha \xi^\beta \wedge \xi^\gamma$, where $c_{\beta\gamma}^\alpha$ are smooth function on U , then $[\xi_\beta, \xi_\gamma] = -c_{\beta\gamma}^\alpha \xi_\alpha$. So we have

$$\begin{aligned}
(d\Phi_l)_p &= \sum_{i_1, \dots, i_{p_l}} \sum_{s=1}^{p_l} (-1)^{s-1} \Phi_{i_1, \dots, i_{p_l}}^{(l)} (\xi^{i_1})_p \wedge \dots \wedge (d\xi^{i_s})_p \wedge \dots \wedge (\xi^{i_{p_l}})_p \\
&= \sum_{i_1, \dots, i_{p_l}} \sum_{s=1}^{p_l} \sum_{\beta, \gamma} \Phi_{i_1, \dots, i_{p_l}}^{(l)} \{c_{\beta\gamma}^{i_s} \xi^\beta \wedge \xi^\gamma \wedge \iota_{\xi_{i_s}} (\xi^{i_1} \wedge \dots \wedge \xi^{i_{p_l}})\}_p \\
&= \sum_{i_1, \dots, i_{p_l}} \sum_{\alpha, \beta, \gamma} \Phi_{i_1, \dots, i_{p_l}}^{(l)} \{c_{\beta\gamma}^\alpha \xi^\beta \wedge \xi^\gamma \wedge \iota_{\xi_\alpha} (\xi^{i_1} \wedge \dots \wedge \xi^{i_{p_l}})\}_p \\
&= A_\Phi^2 \left(\sum_{\alpha, \beta, \gamma} c_{\beta\gamma}^\alpha \xi^\beta \wedge \xi^\gamma \otimes \xi_\alpha \right)_p
\end{aligned}$$

Since we may write

$$\begin{aligned}
T^\nabla(\xi_\beta, \xi_\gamma)_p &= -[\xi_\beta, \xi_\gamma]_p \\
&= -c_{\beta\gamma}^\alpha (\xi_\alpha)_p,
\end{aligned}$$

then we have $d\Phi_p = A_\Phi^2(T^\nabla)_p$ for any $p \in M$ \square

Note that Lie group G acts on $\mathfrak{g} := \text{Lie}(G) \subset \text{End}(V) = V^* \otimes V$ by the adjoint action. Let $v_1, v_2, \dots, v_N \in V$ be a basis and $v^1, v^2, \dots, v^N \in V$ be its dual basis. For each $Q \in \Gamma(R_G(M))$, there is a sub vectorbundle $\hat{\mathfrak{g}}_Q^k := \tilde{Q} \times_G \mathfrak{g}^k$ of $\Lambda^k T^*M \otimes TM$, where

$$\mathfrak{g}^k := \text{span} \left\{ \sum_{i,j} (\alpha \wedge a_i^j v^i) \otimes v_j; \alpha \in \Lambda^{k-1}, \sum_{i,j} a_i^j v^i \otimes v_j \in \mathfrak{g} \right\} \subset \Lambda^k \otimes V$$

for $k \geq 2$, and

$$\mathfrak{g}^1 := \mathfrak{g}, \quad \mathfrak{g}^0 := \{0\}.$$

Then we have an orthogonal decomposition $\Lambda^k \otimes V = \mathfrak{g}^k \oplus P_{\mathfrak{g}}^k$ where $P_{\mathfrak{g}}^k$ is the orthogonal complement. If we put $\hat{P}_Q^k := \tilde{Q} \times_G P_{\mathfrak{g}}^k$, we have an orthogonal decomposition

$$\Lambda^k T^*M \otimes TM = \hat{\mathfrak{g}}_Q^k \oplus \hat{P}_Q^k$$

with respect to g_Q .

Let $\bar{A}_{\Phi_V}^k := A_{\Phi_V}^k|_{P_{\mathfrak{g}}^k}$. It is clear that \mathfrak{g}^k is a subspace of $\text{Ker}(A_{\Phi_V}^k)$ from the definitions of \mathfrak{g}^k and $A_{\Phi_V}^k$. If we assume that $\dim E_{\Phi_V}^k = \dim P_{\mathfrak{g}}^k$, then the induced bundle map $\bar{A}_\Phi^k : \hat{P}_Q^k \rightarrow E_\Phi^k$ is an isomorphism for each $\Phi \in \Gamma(R_{\Phi_V}(M))$.

Proposition 2.4 ([15]). *We define a linear map $\mathbf{a} : V^* \otimes \text{End}(V) \rightarrow \Lambda^2 \otimes V$ by*

$$\mathbf{a}(u_1 \otimes u_2 \otimes v) := u_1 \wedge u_2 \otimes v = (u_1 \otimes u_2 - u_2 \otimes u_1) \otimes v$$

for $u_1, \in V^*$, $u_2 \otimes v \in V^* \otimes V = \text{End}(V)$.

Then $\mathbf{a}|_{V^* \otimes \mathfrak{so}(N)} : V^* \otimes \mathfrak{so}(N) \rightarrow \Lambda^2 \otimes V$ is an isomorphism, where $\mathfrak{so}(N)$ is the Lie algebra of $O(N)$.

See Proposition 2.1 of [15] as to the proof.

Proposition 2.5. *Let $\Phi \in \Gamma(R_{\Phi V}(M))$ and suppose $\dim E_{\Phi V}^2 = \dim P_{\mathfrak{g}}^2$. Then Q_Φ is a torsion-free G -structure if $d\Phi = 0$.*

Proof. Let ∇ be a connection on Q_Φ and assume that $d\Phi = 0$. We may write $\nabla = \nabla^\Phi + \gamma$, where γ is a section of $\widetilde{Q_\Phi} \times_G (V^* \otimes \mathfrak{so}(N))$ and ∇^Φ is the Levi-Civita connection of $g_\Phi := g_{Q_\Phi}$. Then

$$T^\nabla = T^{\nabla^\Phi} + \mathbf{a}(\gamma) = \mathbf{a}(\gamma).$$

Since we have $A_\Phi^2(T^\nabla) = d\Phi = 0$ from the assumption and Proposition 2.3, then

$$\mathbf{a}(\gamma) = T^\nabla \in \text{Ker}(A_\Phi^2) = \Gamma(\mathfrak{g}_{Q_\Phi}^2).$$

Then γ is a section of $\widetilde{Q_\Phi} \times_G (V^* \otimes \mathfrak{g})$ from Proposition 2.4, which means $\gamma \in \Omega^1(\mathfrak{g}_{Q_\Phi}^1)$. Hence we have shown that the Levi-Civita connection $\nabla^\Phi = \nabla - \gamma$ is a connection on Q_Φ . \square

Theorem 2.6. *Let ∇ be the Levi-Civita connection of g_Φ for a section $\Phi \in \Gamma(R_{\Phi V}(M))$. We suppose $\dim E_{\Phi V}^2 = \dim P_{\mathfrak{g}}^2$. Then the following conditions are equivalent.*

(i) $d\Phi = 0$. (ii) Q_Φ is a torsion-free G -structure. (iii) $\nabla\Phi = 0$.

Proof. Proposition 2.5 gives (i) \implies (ii).

Assume that Q_Φ is a torsion-free G -structure. Then the Levi-Civita connection ∇ is a connection on Q_Φ . If we take $p \in M$, U and ξ^1, \dots, ξ^n as in Proposition 2.3, then $(\nabla \xi^i)_p = 0$. So

$$\begin{aligned} (\nabla \Phi_i)_p &= \sum_{j_1, \dots, j_{p_i}} \sum_{s=1}^{p_i} \Phi_{j_1, \dots, j_{p_i}}^i (\xi^{j_1})_p \wedge \dots \wedge (\nabla \xi^{j_s})_p \wedge \dots \wedge (\xi^{j_{p_i}})_p \\ &= 0. \end{aligned}$$

Thus we have shown $\nabla\Phi = 0$ if Q_Φ is a torsion-free G -structure.

If we assume $\nabla\Phi = 0$, then $d\Phi = \sum_j \xi^j \wedge \nabla_{\xi_j} \Phi = 0$. \square

Next we consider the deformation complex of $\Phi_0 \in \widetilde{\mathcal{M}}_{\Phi^V}(M)$.

Proposition 2.7 ([5]). *Let $\Phi_0 \in \widetilde{\mathcal{M}}_{\Phi^V}(M)$. Then $d\Gamma(E_{\Phi_0}^k)$ is a subspace of $\Gamma(E_{\Phi_0}^{k+1})$.*

From Proposition 2.7, we obtain Goto's complex

$$\cdots \xrightarrow{d} \Gamma(E_{\Phi_0}^k) \xrightarrow{d} \Gamma(E_{\Phi_0}^{k+1}) \xrightarrow{d} \cdots. \quad (1)$$

3 Moduli spaces of the quaternionic Kähler structures

In this section, we state the main result in this paper. First, we define quaternionic Kähler structures and their moduli space.

From now on we consider the case of $N = 4n$. We put $I, J, K \in M_{4n}\mathbb{R}$ be almost complex structures on V defined by

$$I = \begin{pmatrix} O & -E_n & O & O \\ E_n & O & O & O \\ O & O & O & -E_n \\ O & O & E_n & O \end{pmatrix} \quad J = \begin{pmatrix} O & O & -E_n & O \\ O & O & O & E_n \\ E_n & O & O & O \\ O & -E_n & O & O \end{pmatrix}$$

$$K = \begin{pmatrix} O & O & O & -E_n \\ O & O & -E_n & O \\ O & E_n & O & O \\ E_n & O & O & O \end{pmatrix}$$

where E_n is the unit matrix of $GL_n\mathbb{R}$, and set

$$\omega_I := g_0(I \cdot, \cdot), \quad \omega_J := g_0(J \cdot, \cdot), \quad \omega_K := g_0(K \cdot, \cdot).$$

Then we have an 4-form

$$\Phi_{qK} := \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K \in \Lambda^4,$$

whose isotropy group

$$G_{\Phi_{qK}} := \{g \in GL_N\mathbb{R}; \rho(g)\Phi_{qK} = \Phi_{qK}\}$$

is equal to $Sp(n)Sp(1) := Sp(n) \times_{\{\pm 1\}} Sp(1)$.

Definition 3.1. Let M be a smooth manifold of dimension $4n$. Then we call $\Phi_0 \in \Gamma(R_{\Phi_{qK}}(M))$ is a quaternionic Kähler structure on M if and only if $d\Phi_0 = 0$.

There is the irreducible decompositions of $Sp(n)Sp(1)$ -representation according to [3][15],

$$\begin{aligned}\Lambda^3 \otimes \mathbb{C} &= \lambda_0^3 \sigma^3 \oplus \lambda_0^1 \sigma^3 \oplus \lambda_1^3 \sigma^1 \oplus \lambda_0^1 \sigma^1, \\ \Lambda^4 \otimes \mathbb{C} &= \lambda_0^4 \sigma^4 \oplus \lambda_0^2 \sigma^4 \oplus \sigma^4 \oplus \lambda_1^4 \sigma^2 \oplus \lambda_1^2 \sigma^2 \oplus \lambda_0^2 \sigma^2 \oplus \lambda_2^4 \oplus \lambda_0^2 \oplus \sigma^0, \\ \Lambda^5 \otimes \mathbb{C} &= \lambda_0^5 \sigma^5 \oplus \lambda_0^3 \sigma^5 \oplus \lambda_0^1 \sigma^5 \oplus \lambda_1^5 \sigma^3 \oplus \lambda_1^3 \sigma^3 \oplus \lambda_2^5 \sigma^1 \oplus \lambda_0^3 \sigma^1 \oplus \Lambda^3 \otimes \mathbb{C}.\end{aligned}$$

Here we write $\lambda_q^p \sigma^r = \lambda_q^p \otimes \sigma^r$ where λ_q^p is an irreducible $Sp(n)$ -module and $\sigma^r = S^r(\mathbb{C}^2)$ is an irreducible $Sp(1)$ -module. The representation λ_q^p has the highest weight (μ_1, \dots, μ_n) such that

$$\mu_l = \begin{cases} 2 & 1 \leq l \leq q, \\ 1 & q+1 \leq l \leq p-q, \\ 0 & p-q+1 \leq l \leq n. \end{cases}$$

Then we have

$$\begin{aligned}E_{\Phi_{qK}}^0 \otimes \mathbb{C} &= \lambda_0^1 \sigma^1, \\ E_{\Phi_{qK}}^1 \otimes \mathbb{C} &= \lambda_0^2 \sigma^2 \oplus \lambda_1^2 \sigma^2 \oplus \lambda_0^2 \oplus \sigma^0,\end{aligned}$$

by the definition of Φ_{qK} and direct calculation. As to $E_{\Phi_{qK}}^2$, there is the irreducible decomposition for $n \geq 3$

$$E_{\Phi_{qK}}^2 \otimes \mathbb{C} = \lambda_1^3 \sigma^3 \oplus \lambda_0^3 \sigma^1 \oplus \Lambda^3 \otimes \mathbb{C}.$$

by [19].

Weyl dimension formula [4] of $Sp(n)$ representation gives

$$\dim_{\mathbb{C}} \lambda_q^p = \frac{2^n n! \prod_{1 \leq i, j \leq n} (\tilde{\mu}_i - \tilde{\mu}_j) (\tilde{\mu}_i + \tilde{\mu}_j + 2n + 2) \prod_{k=1}^n (\tilde{\mu}_k + n + 1)}{\prod_{k=1}^n (2k)!}.$$

Then we can calculate the dimension of $E_{\Phi_{qK}}^2$.

Theorem 3.2 ([19]). *Let M be a $4n$ -dimensional manifold for $n \geq 3$ and $\Phi_0 \in \Gamma(R_{\Phi_{qK}}(M))$. Then the Levi-Civita connection ∇^{Φ_0} of g_{Φ_0} reduces to the connection of the principal $Sp(n)Sp(1)$ -bundle Q_{Φ_0} if and only if $d\Phi_0 = 0$.*

Proof. It suffices to show that $\dim E_{\Phi_{qK}}^2 = \dim P_{\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)}^2$ from Theorem 2.6. By Weyl dimension formula, we have

$$\dim E_{\Phi_{qK}}^2 = 24n^3 - 12n^2 - 12n,$$

for $n \geq 3$. From $\dim(\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)) = 2n^2 + n + 3$ and the injectivity of $\mathbf{a}|_{V^* \otimes \mathfrak{so}(4n)}$, we have

$$\begin{aligned}\dim P_{\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)}^2 &= \dim(\Lambda^2 \otimes V) - \dim\{V^* \otimes (\mathfrak{sp}(n) \oplus \mathfrak{sp}(1))\} \\ &= 24n^3 - 12n^2 - 12n.\end{aligned}$$

□

Now we denote the scalar curvature of a Riemannian metric g by κ_g . If the holonomy group $Hol(g)$ is isomorphic to the subgroup of $Sp(n)Sp(1)$, then g is Einstein, so κ_g is constant [15]. Moreover $Hol(g)$ is isomorphic to $Sp(n)Sp(1)$ if and only if $\kappa_{g_{\Phi_0}} \neq 0$. So we put

$$\widetilde{\mathcal{M}}_{qK} := \{\Phi_0 \in \widetilde{\mathcal{M}}_{\Phi_{qK}}(M); \kappa_{g_{\Phi_0}} \neq 0\}.$$

Let $\mathcal{G} := \text{Diff}_0(M) \times \mathbb{R}_{>0}$ where $\text{Diff}_0(M)$ is the identity component of $\text{Diff}(M)$. Then \mathcal{G} acts on $\widetilde{\mathcal{M}}_{qK}$ by putting $(f, c) \cdot \Phi := cf^*\Phi$ for $(f, c) \in \mathcal{G}$. Thus we obtain the moduli spaces of quaternionic Kähler structures of nonzero scalar curvature

$$\mathcal{M}_{qK} := \widetilde{\mathcal{M}}_{qK} / \mathcal{G}.$$

Next we show the rigidity theorem for quaternionic Kähler structures of nonzero scalar curvature. From now on, we suppose that M is compact. We will use following lemmas.

Lemma 3.3. *Let M be a compact manifold of dimension $4n \geq 12$. Then Goto's complex of quaternionic Kähler structures*

$$\cdots \xrightarrow{d} \Gamma(E_{\Phi_0}^k) \xrightarrow{d} \Gamma(E_{\Phi_0}^{k+1}) \xrightarrow{d} \cdots$$

is elliptic complex at $k = 1$ for each $\Phi_0 \in \widetilde{\mathcal{M}}_{\Phi_{qK}}(M)$. In particular, there is the Hodge decomposition

$$\Gamma(E_{\Phi_0}^1) = \mathbb{H}_{\Phi_0}^1 \oplus d\Gamma(E_{\Phi_0}^0) \oplus d_1^*\Gamma(E_{\Phi_0}^2),$$

where d_k^ is a formal adjoint operator of $d : \Gamma(E_{\Phi_0}^k) \rightarrow \Gamma(E_{\Phi_0}^{k+1})$ and $\mathbb{H}_{\Phi_0}^1$ is given by*

$$\mathbb{H}_{\Phi_0}^1 := \text{Ker}(\Delta_{\sharp} := dd_0^* + d_1^*d : \Gamma(E_{\Phi_0}^1) \rightarrow \Gamma(E_{\Phi_0}^1)).$$

Lemma 3.4. *Let M be a compact manifold of dimension $4n \geq 12$. Then we have*

$$\mathbb{H}_{\Phi_0}^1 = \mathbb{R}\Phi_0$$

for each $\Phi_0 \in \widetilde{\mathcal{M}}_{qK}$.

We will prove Lemma 3.3 and 3.4 in Section 4 and 5, respectively. Let F be a fibre bundle over M and $k \geq 2n + 1$. Then an L_k^2 -section of F is a C^0 -section by Sobolev embedding theorem. By putting $(f, c) \cdot \Phi := cf^*\Phi \in$

$L_k^2(R_{\Phi_{qK}}(M))$ for $(f, c) \in L_{k+1}^2(\text{Diff}_0(M)) \times \mathbb{R}_{>0}$ and $\Phi \in L_k^2(R_{\Phi_{qK}}(M))$, an infinite dimensional Lie group $\mathcal{G}_{k+1} := L_{k+1}^2(\text{Diff}_0(M)) \times \mathbb{R}_{>0}$ acts on a Hilbert manifold $L_k^2(R_{\Phi_{qK}}(M))$. Thus we have a quotient topological space

$$\mathcal{A}_k := L_k^2(R_{\Phi_{qK}}(M))/\mathcal{G}_{k+1}$$

and the quotient map $\pi_k : L_k^2(R_{\Phi_{qK}}(M)) \rightarrow \mathcal{A}_k$. Then we are going to show that $\pi_k(\widetilde{\mathcal{M}}_{qK})$ is a discrete subset of \mathcal{A}_k for $k \geq 2n + 1$ and $n \geq 3$. This is proven directly from Proposition 3.7.

First we consider the neighborhood of $\pi_k(\Phi_0) \in \mathcal{A}_k$ for $\Phi_0 \in \widetilde{\mathcal{M}}_{\Phi_{qK}}(M)$. Since $\bar{A}_{\Phi_0}^1 : \hat{P}_{Q_{\Phi_0}}^1 \rightarrow E_{\Phi_0}^1$ is an isomorphism, there is the inverse map $(\bar{A}_{\Phi_0}^1)^{-1}$. Then we define a map $\varphi_{k, \Phi_0} : L_k^2(E_{\Phi_0}^1) \rightarrow L_k^2(R_{\Phi_{qK}}(M))$ by $\varphi_{k, \Phi_0}(\alpha) := \rho(e^{(\bar{A}_{\Phi_0}^1)^{-1}\alpha})\Phi_0$ for $\Phi_0 \in \widetilde{\mathcal{M}}_{\Phi_{qK}}(M)$ and $\alpha \in L_k^2(E_{\Phi_0}^1)$, where $\rho(g)\Phi_0 = (g^{-1})^*\Phi_0$ for $g \in L_k^2(GL(TM))$ and $e^X := \sum_{j=0}^{\infty} X^j/j!$ for $X \in L_k^2(\text{End } TM)$. The differential of the map φ_{k, Φ_0} at the origin is given by

$$(\varphi_{k, \Phi_0})_0(\beta) = -\bar{A}_{\Phi_0}^1(\bar{A}_{\Phi_0}^1)^{-1}(\beta) = -\beta$$

for $\beta \in L_k^2(E_{\Phi_0}^1)$. If we put

$$U_{k, \Phi_0}(\varepsilon) := \{\alpha \in L_k^2(E_{\Phi_0}^1); \|\alpha\|_{L_k^2} < \varepsilon\}$$

for $\varepsilon > 0$, then there is $\varepsilon > 0$ such that

$$\varphi_{k, \Phi_0}|_{U_{k, \Phi_0}(\varepsilon)} : U_{k, \Phi_0}(\varepsilon) \longrightarrow \varphi_{k, \Phi_0}(U_{k, \Phi_0}(\varepsilon))$$

is a diffeomorphism from inverse function theorem.

Set $V_{k, \Phi_0}(\varepsilon) := \{\alpha \in U_{k, \Phi_0}(\varepsilon); d_0^*\alpha = 0, \langle \alpha, \Phi_0 \rangle_{L^2(\Phi_0)} = 0\}$ where $\langle \alpha, \Phi_0 \rangle_{L^2(\Phi_0)} = \int_M g_{\Phi_0}(\alpha, \Phi_0) \text{vol}_{g_{\Phi_0}}$.

Lemma 3.5. *Let $\Phi_0 \in \widetilde{\mathcal{M}}_{\Phi_{qK}}(M)$ and $k \geq 2n + 1$. Then there is an open neighborhood $W_{k, \Phi_0} \subset \mathcal{A}_k$ of $\pi_k(\Phi_0)$ which satisfies $W_{k, \Phi_0} \subset \pi_k \circ \varphi_{k, \Phi_0}(V_{k, \Phi_0}(\varepsilon))$ for any $\varepsilon > 0$.*

Proof. Let the map

$$F : \varphi_{k, \Phi_0}(V_{k, \Phi_0}(\varepsilon)) \times L_{k+1}^2(\text{Diff}_0(M)) \times \mathbb{R}_{>0} \longrightarrow L_k^2(R_{\Phi_{qK}}(M))$$

be given by $F(\Phi, f, c) := cf^*\Phi$ for $\Phi \in \varphi_{k, \Phi_0}(V_{k, \Phi_0}(\varepsilon))$, $f \in L_{k+1}^2(\text{Diff}_0(M))$ and $c \in \mathbb{R}_{>0}$. We take

$$\begin{aligned} (\alpha, X, t) &\in V_{k, \Phi_0} \oplus L_{k+1}^2(TM) \oplus \mathbb{R} \\ &= T_{(\Phi_0, Id_M, 1)}\{\varphi_{k, \Phi_0}(V_{k, \Phi_0}(\varepsilon)) \times L_{k+1}^2(\text{Diff}_0(M)) \times \mathbb{R}_{>0}\} \end{aligned}$$

where $V_{k,\Phi_0} := \{\alpha \in L_k^2(E_{\Phi_0}^1); d_0^*\alpha = 0, \langle \alpha, \Phi_0 \rangle_{L^2(\Phi_0)} = 0\}$. Then the differential of F at $(\Phi_0, Id_M, 1)$ is given by

$$F_*|_{(\Phi_0, Id_M, 1)}(\alpha, X, t) = t\Phi_0 + d\iota_X\Phi_0 + \alpha.$$

Thus $F_*|_{(\Phi_0, Id_M, 1)} : V_{k,\Phi_0} \oplus L_{k+1}^2(TM) \oplus \mathbb{R} \rightarrow L_k^2(E_{\Phi_0}^1)$ is an isomorphism by Lemma 3.3, so there are some $\delta > 0$ and neighborhood $N_{(Id_M, 1)} \subset L_{k+1}^2(\text{Diff}_0(M)) \times \mathbb{R}_{>0} = \mathcal{G}_{k+1}$ of $(Id_M, 1)$ such that $F|_{\varphi_{k,\Phi_0}(V_{k,\Phi_0}(\delta)) \times N_{(Id_M, 1)}}$ is a diffeomorphism. In particular, $F(\varphi_{k,\Phi_0}(V_{k,\Phi_0}(\delta)) \times N_{(Id_M, 1)})$ is an open set of $L_k^2(R_{\Phi_{qK}}(M))$. Hence by putting

$$W_{k,\Phi_0} := \pi_k \circ F(\varphi_{k,\Phi_0}(V_{k,\Phi_0}(\delta')) \times N_{(Id_M, 1)})$$

for $\delta' = \min\{\delta, \varepsilon\}$, we have

$$W_{k,\Phi_0} \subset \pi_k \circ F(\varphi_{k,\Phi_0}(V_{k,\Phi_0}(\varepsilon)) \times N_{(Id_M, 1)}) = \pi_k \circ \varphi_{k,\Phi_0}(V_{k,\Phi_0}(\varepsilon)).$$

□

Proposition 3.6. *Let $\Phi_0 \in \widetilde{\mathcal{M}}_{qK}$ and $n \geq 3$. Then there is $\varepsilon > 0$ which satisfies the following condition. If $\Phi \in \varphi_{2n+1,\Phi_0}(V_{2n+1,\Phi_0}(\varepsilon))$ satisfies $d\Phi = 0$, then $\Phi = \Phi_0$.*

Proof. Fix $\varepsilon > 0$ and $\Phi \in \varphi_{2n+1,\Phi_0}(V_{2n+1,\Phi_0}(\varepsilon))$. Then we may write $\Phi = \rho(e^a)\Phi_0$ for $a \in L_{2n+1}^2(\hat{P}_{Q_{\Phi_0}}^1)$ which satisfies $\bar{A}_{\Phi_0}^1(a) \in V_{2n+1,\Phi_0}(\varepsilon)$. If we put

$$\rho_*(a)\beta := \left. \frac{d}{dt} \right|_{t=0} (e^{-ta})^* \beta$$

for $\beta \in \Omega(M)$, then it follows $\rho_*(a)\Phi_0 = -\bar{A}_{\Phi_0}^1(a)$ and

$$\rho(e^a)\Phi_0 = \sum_{j=0}^{\infty} \frac{1}{j!} \{\rho_*(a)\}^j \Phi_0.$$

Then

$$\begin{aligned} d\Phi &= d \sum_{j=0}^{\infty} \frac{1}{j!} \{\rho_*(a)\}^j \Phi_0 \\ &= -d\bar{A}_{\Phi_0}^1(a) + \sum_{j=2}^{\infty} \frac{1}{j!} d\{\rho_*(a)\}^j \Phi_0. \end{aligned} \tag{2}$$

Since the linear operators

$$\begin{aligned} \rho_* &: L_{2n+1}^2(\text{End}(TM)) \longrightarrow L_{2n+1}^2(\text{End}(\Lambda^4 T^* M)), \\ d &: L_{2n+1}^2(\Lambda^4 T^* M) \longrightarrow L_{2n}^2(\Lambda^5 T^* M) \end{aligned}$$

are the bounded operators, there are constants $s, K_0, K_1 > 0$ depending only on M and Φ_0 , such that

$$\begin{aligned}\|\{\rho_*(a)\}^j \Phi_0\|_{L_{2n+1}^2} &\leq K_0 s^j \|a\|_{L_{2n+1}^2}^j \|\Phi_0\|_{L_{2n+1}^2}, \\ \|d\beta\|_{L_{2n}^2} &\leq K_1 \|\beta\|_{L_{2n+1}^2}\end{aligned}$$

for $a \in L_{2n+1}^2(\hat{P}_{Q_{\Phi_0}}^1)$ and $\beta \in L_{2n+1}^2(\Lambda^4 T^* M)$. So we have

$$\begin{aligned}\left\| \sum_{j=2}^{\infty} \frac{1}{j!} d\{\rho_*(a)\}^j \Phi_0 \right\|_{L_{2n}^2} &\leq \sum_{j=2}^{\infty} \frac{1}{j!} \|d\{\rho_*(a)\}^j \Phi_0\|_{L_{2n}^2} \\ &\leq K_0 K_1 \sum_{j=2}^{\infty} \frac{1}{j!} (s \|a\|_{L_{2n+1}^2})^j \|\Phi_0\|_{L_{2n+1}^2} \\ &= K_0 K_1 s^2 \|\Phi_0\|_{L_{2n+1}^2} \|a\|_{L_{2n+1}^2}^2 f(s \|a\|_{L_{2n+1}^2}),\end{aligned}$$

where a C^∞ -function $f : \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$f(x) = \begin{cases} (e^x - 1 - x)/x^2 & (x \neq 0), \\ 1/2 & (x = 0). \end{cases}$$

If we take $\varepsilon \leq (K_2 s)^{-1}$ where K_2 is the operator norm of the bounded operator

$$(\bar{A}_{\Phi_0}^1)^{-1} : L_{2n+1}^2(E_{\Phi_0}^1) \longrightarrow L_{2n+1}^2(\hat{P}_{Q_{\Phi_0}}^1),$$

then we have

$$f(s \|a\|_{L_{2n+1}^2}) \leq f_{\max} := \max\{f(x); x \in [0, 1]\} < \infty.$$

From now on, we suppose $d\Phi = 0$. Then (2) gives

$$d\bar{A}_{\Phi_0}^1(a) = \sum_{j=2}^{\infty} \frac{1}{j!} d\{\rho_*(a)\}^j \Phi_0,$$

so it follows

$$\begin{aligned}\|d\bar{A}_{\Phi_0}^1(a)\|_{L_{2n}^2} &= \left\| \sum_{j=2}^{\infty} \frac{1}{j!} d\{\rho_*(a)\}^j \Phi_0 \right\|_{L_{2n}^2} \\ &\leq K_0 K_1 f_{\max} s^2 \|\Phi_0\|_{L_{2n+1}^2} \|a\|_{L_{2n+1}^2}^2.\end{aligned}\tag{3}$$

From Lemma 3.3 and Lemma 3.4, there is the decomposition

$$\begin{aligned}L_{2n+1}^2(E_{\Phi_0}^1) &= \mathbb{R}\Phi_0 \oplus dL_{2n+2}^2(E_{\Phi_0}^0) \oplus d_1^* L_{2n+2}^2(E_{\Phi_0}^2) \\ &= \mathbb{R}\Phi_0 \oplus L_{2n+1}^2(Im(\Delta_{\sharp})).\end{aligned}$$

Then $\bar{A}_{\Phi_0}^1(a)$ is an element of $d_1^* L_{2n+2}^2(E_{\Phi_0}^2)$ since $\bar{A}_{\Phi_0}^1(a) \in V_{2n+1, \Phi_0}(\varepsilon)$. From Lemma 3.3, there is the Green operator

$$G_{\sharp} : L_{2n-1}^2(Im(\Delta_{\sharp})) \longrightarrow L_{2n+1}^2(Im(\Delta_{\sharp})),$$

which is the inverse operator of $\Delta_{\sharp}|_{L_{2n+1}^2(Im(\Delta_{\sharp}))}$. Since $d_0^* \bar{A}_{\Phi_0}^1(a) = 0$, we have

$$\begin{aligned} \|\bar{A}_{\Phi_0}^1(a)\|_{L_{2n+1}^2} &= \|G_{\sharp} \Delta_{\sharp} \bar{A}_{\Phi_0}^1(a)\|_{L_{2n+1}^2} \\ &\leq K_3 K_4 \|d \bar{A}_{\Phi_0}^1(a)\|_{L_{2n}^2}, \end{aligned} \quad (4)$$

where K_3 and K_4 are the operator norms of G_{\sharp} and d_1^* , respectively. Thus we have an inequality

$$\|a\|_{L_{2n+1}^2} \leq K_0 K_1 K_2 K_3 K_4 s^2 f_{max} \|\Phi_0\|_{L_{2n+1}^2} \|a\|_{L_{2n+1}^2}^2,$$

from (3) and (4). Then we obtain an estimate

$$\|a\|_{L_{2n+1}^2} (1 - K_0 K_1 K_2 K_3 K_4 s^2 f_{max} \|\Phi_0\|_{L_{2n+1}^2} \|a\|_{L_{2n+1}^2}) \leq 0.$$

So if we put $\varepsilon = \min\{(K_2 s)^{-1}, (2K_0 K_1 K_2 K_3 K_4 s^2 f_{max} \|\Phi_0\|_{L_{2n+1}^2})^{-1}\} > 0$, then it follows $\|a\|_{L_{2n+1}^2} = 0$, which means $\Phi = \Phi_0$. \square

Proposition 3.7. *Let $\Phi_0 \in \widetilde{\mathcal{M}}_{qK}$ and $n \geq 3$. Then the set $\{\pi_k(\Phi_0)\}$ is an open subset of $\pi_k(\widetilde{\mathcal{M}}_{qK})$ for $k \geq 2n+1$.*

Proof. Take $\varepsilon > 0$ as in Proposition 3.6. We can take an open set W_{k, Φ_0} of \mathcal{A}_k such that $W_{k, \Phi_0} \subset \pi_k \circ \varphi_{k, \Phi_0}(V_{k, \Phi_0}(\varepsilon))$ from Lemma 3.5. Then $W_{k, \Phi_0} \cap \pi_k(\widetilde{\mathcal{M}}_{qK})$ is an open set of $\pi_k(\widetilde{\mathcal{M}}_{qK})$. Let x be an element of $W_{k, \Phi_0} \cap \pi_k(\widetilde{\mathcal{M}}_{qK})$. Then we may write $x = \pi_k(\Phi)$ for some $\Phi \in \varphi_{k, \Phi_0}(V_{k, \Phi_0}(\varepsilon))$, but Proposition 3.6 gives $\Phi = \Phi_0$ since $\varphi_{k, \Phi_0}(V_{k, \Phi_0}(\varepsilon)) \subset \varphi_{2n+1, \Phi_0}(V_{2n+1, \Phi_0}(\varepsilon))$. Hence we have $W_{k, \Phi_0} \cap \pi_k(\widetilde{\mathcal{M}}_{qK}) = \{\pi_k(\Phi_0)\}$. \square

Thus we have shown that $\pi_k(\widetilde{\mathcal{M}}_{qK})$ is a discrete subset of \mathcal{A}^k for $k \geq 2n+1$ and $n \geq 3$. Next we are going to prove Theorem 1.3.

Lemma 3.8. *Let $\Phi_A, \Phi_B \in \widetilde{\mathcal{M}}_{\Phi_{qK}}(M)$ and $N \geq 2n+1$. Suppose that $\pi_k(\Phi_A) = \pi_k(\Phi_B)$ for any $k \geq N$. Then there is $(f, c) \in \mathcal{G} = \text{Diff}_0(M) \times \mathbb{R}_{>0}$ such that $c f^* \Phi_A = \Phi_B$.*

Proof. Suppose that we have $\pi_k(\Phi_A) = \pi_k(\Phi_B)$ for any $k \geq N$. Then there is $(f_k, c_k) \in \mathcal{G}_{k+1}$ such that $c_k f_k^* \Phi_A = \Phi_B$ for each k , so we have $c_N f_N^* \Phi_A = c_k f_k^* \Phi_A$. Since each f_k is homotopic to Id_M , we have $[c_k f_k^* \Phi_A]_{dR} = [c_k \Phi_A]_{dR}$

where $[\theta]_{dR} \in H_{dR}^4(M)$ is the de Rham class of a closed form $\theta \in \Omega^4$. Then if we denote by Π_h the harmonic projection with respect to g_{Φ_A} , we have

$$\Pi_h(c_k f_k^* \Phi_A) = \Pi_h(c_k \Phi_A) = c_k \Phi_A.$$

Hence it follows that $c_k \Phi_A = c_N \Phi_A$, which gives $c_k = c_N$. Then $f_N \circ f_k^{-1} \in L_{N+1}^2(\text{Diff}_0(M))$ is an element of

$$I_{\Phi_A} := \{f \in L_{N+1}^2(\text{Diff}_0(M)); f^* \Phi_A = \Phi_A\}.$$

According to [13], if a C^1 diffeomorphism $f : M \rightarrow M$ preserves a smooth Riemannian metric, then f is smooth. Since each element of I_{Φ_A} preserves smooth Riemannian metric g_{Φ_A} , then I_{Φ_A} is a subgroup of $\text{Diff}_0(M)$. So $\tilde{f}_k := f_N \circ f_k^{-1}$ is an element of $\text{Diff}_0(M)$, then we may write $f_N = \tilde{f}_k \circ f_k$, which is an element of $L_{k+1}^2(\text{Diff}_0(M))$. Thus f_N is an element of

$$\bigcap_{k=N}^{\infty} L_{k+1}^2(\text{Diff}_0(M)) = \text{Diff}_0(M).$$

□

Now we have a quotient space $\mathcal{M}_{qK} := \widetilde{\mathcal{M}}_{qK}/\mathcal{G}$ and the quotient map

$$\pi_{qK} : \widetilde{\mathcal{M}}_{qK} \longrightarrow \mathcal{M}_{qK}.$$

Definition 3.9. Quaternionic Kähler structures $\{\Phi_t\}_{t \in \mathbb{R}} \subset \widetilde{\mathcal{M}}_{qK}$ is a continuous family if the map

$$\widetilde{\Phi}_k : \mathbb{R} \longrightarrow L_k^2(R_{\Phi_{qK}}(M))$$

defined by $\widetilde{\Phi}_k(t) := \Phi_t$ is a continuous map for each $k \geq 2n + 1$.

Then a rigidity theorem for quaternionic Kähler structures is obtained as follows.

Theorem 3.10. *Let $\{\Phi_t\}_{t \in \mathbb{R}} \subset \widetilde{\mathcal{M}}_{qK}$ be a continuous family on a compact manifold M of dimension $4n$ for $n \geq 3$. Then we have $\pi_{qK}(\Phi_t) = \pi_{qK}(\Phi_0)$ for any $t \in \mathbb{R}$.*

Proof. Since the maps $\pi_k \circ \widetilde{\Phi}_k$ are continuous maps, we have $\pi_k(\Phi_t) = \pi_k(\Phi_0)$ from Proposition 3.7. Then by Lemma 3.8, we obtain $\pi_{qK}(\Phi_t) = \pi_{qK}(\Phi_0)$. □

4 Deformation complexes of torsion-free G -structures

The purpose of this section is to give the proof of Lemma 3.3. We introduce new complexes; the deformation complexes of torsion-free G -structures. The new complex is elliptic at $k = 1$ if G satisfies a certain condition. Then we can construct an isomorphism between the deformation complex of torsion-free $Sp(n)Sp(1)$ -structures and Goto's complex (1) of quaternionic Kähler structures.

Let G be a Lie subgroup of $O(N)$ and M be a compact manifold of dimension N . Fix a torsion-free G -structure $Q \in \Gamma(R_G(M))$. Let $d_k^\nabla : \Omega^k(TM) \rightarrow \Omega^{k+1}(TM)$ be the covariant exterior derivative of the Levi-Civita connection ∇ of g_Q . Then it is easy to see that $d_k^\nabla \Gamma(\hat{\mathfrak{g}}_Q^k)$ is a subspace of $\Gamma(\hat{\mathfrak{g}}_Q^{k+1})$ for $k = 1, 2, \dots$. Moreover, there is a following property.

Proposition 4.1. *Let $Q \in \Gamma(R_G(M))$ be a torsion-free G -structure, and ∇ be the Levi-Civita connection of g_Q . Then $d_{k+1}^\nabla \circ d_k^\nabla (\Gamma(\Lambda^k T^*M \otimes TM))$ is a subspace of $\Gamma(\hat{\mathfrak{g}}_Q^{k+2})$.*

Proof. First we suppose $k = 0$. We are going to show $(d^\nabla)^2 X \in \Gamma(\hat{\mathfrak{g}}_Q^2)$ for each $X \in \mathcal{X}(M)$.

Fix $p \in M$, and take a neighborhood U of p and local orthonormal frame $\xi_1, \dots, \xi_N \in \mathcal{X}(U)$ and its dual frame $\xi^1, \dots, \xi^N \in \Omega^1(U)$ as in the proof of Proposition 2.3. Let $R \in \Omega^2(\hat{\mathfrak{g}}_Q)$ be a curvature tensor of ∇ , and we may write $R(\xi_i, \xi_j)\xi_l = \sum_m R_{ijl}^m \xi_m$ and $X = \sum_l X^l \xi_l \in \mathcal{X}(U)$ on U . Since the curvature tensor of G -connection is a section of the vector bundle induced by the adjoint action of \mathfrak{g} , so $\sum_{l,m} R_{ijl}^m \xi^l \otimes \xi_m \in \Gamma(T^*U \otimes TU)$ is a section of $\hat{\mathfrak{g}}_Q^1|_U$. Then we have

$$(d^\nabla)^2 X = \sum_{i,j,l,m} R_{ijl}^m X^l \xi^i \wedge \xi^j \otimes \xi_m.$$

So it follows that

$$\begin{aligned} (d^\nabla)^2 X &= \sum_{i,j,l,m} (-R_{jli}^m - R_{lij}^m) X^l \xi^i \wedge \xi^j \otimes \xi_m \\ &= 2 \sum_{j,l} X^l \xi^j \wedge \left(\sum_{i,m} R_{jli}^m \xi^i \otimes \xi_m \right) \in \Gamma(\hat{\mathfrak{g}}_Q^2). \end{aligned}$$

from the first Bianchi identity, $R_{ijl}^m + R_{jli}^m + R_{lij}^m = 0$. Next we show the case of $k \geq 1$. It suffices to show that $(d^\nabla)^2(\alpha \otimes X) \in \Gamma(\hat{\mathfrak{g}}_Q^{k+2})$ for each

$\alpha \in \Omega^k(M)$ and $X \in \mathcal{X}(M)$. Then we have

$$\begin{aligned}
(d^\nabla)^2(\alpha \otimes X) &= d^\nabla(d\alpha \otimes X + (-1)^k \alpha \wedge \nabla X) \\
&= d^2\alpha \otimes X + (-1)^{k+1} d\alpha \wedge \nabla X + (-1)^k d\alpha \wedge \nabla X \\
&\quad + \alpha \wedge (d^\nabla)^2 X \\
&= \alpha \wedge (d^\nabla)^2 X \in \Gamma(\hat{\mathfrak{g}}_Q^{k+2})
\end{aligned}$$

from the case of $k = 0$. \square

We define a differential operator $d_k^Q : \Gamma(\hat{P}_Q^k) \rightarrow \Gamma(\hat{P}_Q^{k+1})$ by $d_k^Q := pr_{\hat{P}_Q^{k+1}} \circ d_k^\nabla$, where $pr_{\hat{P}_Q^k} : \Lambda^k T^*M \otimes TM \rightarrow \hat{P}_Q^k$ is the orthogonal projection. Then from Proposition 4.1, we obtain the deformation complex of torsion-free G -structures

$$\dots \xrightarrow{d_{k-1}^Q} \Gamma(\hat{P}_Q^k) \xrightarrow{d_k^Q} \Gamma(\hat{P}_Q^{k+1}) \xrightarrow{d_{k+1}^Q} \dots \quad (5)$$

Next we will see that the complex (5) is elliptic at $k = 1$. We denote by $Sb_k(u)$ the symbol of the differential operator d_k^Q at $u \in V^* - \{0\}$. Let $pr_{P_{\mathfrak{g}}^k} : \Lambda^k \otimes V \rightarrow P_{\mathfrak{g}}^k$ be the orthogonal projection. Then we have

$$\begin{aligned}
Sb_k(u)(X) &= pr_{P_{\mathfrak{g}}^{k+1}}(u \wedge X) \\
&= pr_{P_{\mathfrak{g}}^{k+1}}\left(\sum_{i_1, \dots, i_k, j} X_{i_1 \dots i_k}^j (u \wedge v^{i_1} \wedge \dots \wedge v^{i_k}) \otimes v_j\right)
\end{aligned}$$

for $X = \sum_{i_1, \dots, i_k, j} X_{i_1 \dots i_k}^j v^{i_1} \wedge \dots \wedge v^{i_k} \otimes v_j \in P_{\mathfrak{g}}^k$. To prove that the complex (5) is elliptic at $k = 1$, we have to see the complex

$$\dots \xrightarrow{Sb_{k-1}(u)} P_{\mathfrak{g}}^k \xrightarrow{Sb_k(u)} P_{\mathfrak{g}}^{k+1} \xrightarrow{Sb_{k+1}(u)} \dots$$

is the exact sequence at $P_{\mathfrak{g}}^1$.

We have an orthogonal decomposition $V = \mathbb{R}v \oplus W_v$ with respect to g_0 for each $v \in V - \{0\}$. The decomposition induces the orthogonal projection $p_v : End(V) \rightarrow End(W_v)$, then we can consider the following conditions for $G \subset O(N)$.

(C1) The linear map $p_v|_{\mathfrak{g}} : \mathfrak{g} \rightarrow End(W_v)$ is injective for each $v \in V - \{0\}$.

The condition **(C1)** is equivalent to the following condition.

(C2) Let $v_1, v_2, \dots, v_N \in V$ be any orthonormal basis and $v^1, v^2, \dots, v^N \in V^*$ be its dual basis. Then for all $A = A_i^j v^i \otimes v_j \in \mathfrak{g}$, $A = 0$ if $A_i^j = 0$ for $i, j \neq 1$.

Lemma 4.2. *Suppose that the Lie subgroup $G \subset O(N)$ satisfies (C1). For all $a \in \text{End}(V)$ and $u \in V^* - \{0\}$, we may write $a = b + u \otimes w$ for some $b \in \mathfrak{g}$ and $w \in V$ if $u \wedge a \in \mathfrak{g}^2$.*

Proof. We may take $u \in V^* - \{0\}$ as $g_0(u, u) = 1$. Then we fix an orthonormal basis $v^1 = u, v^2, \dots, v^N \in V^*$ and its dual basis $v_1, v_2, \dots, v_N \in V$ with respect to g_0 . Suppose that $v^1 \wedge \sum_{j,k} a_j^k v^j \otimes v_k = \sum_{i,j,k} B_{ij}^k v^i \wedge v^j \otimes v_k$ for $a = \sum_{j,k} a_j^k v^j \otimes v_k \in \text{End}(V)$ and $\sum_{j,k} B_{ij}^k v^j \otimes v_k \in \mathfrak{g}$ for $i = 1, 2, \dots, N$. Then

$$v^1 \wedge \sum_{j,k} a_j^k v^j \otimes v_k = \sum_{i < j} \sum_k (B_{ij}^k - B_{ji}^k) v^i \wedge v^j \otimes v_k.$$

So we have

$$\begin{aligned} a_j^k &= B_{1j}^k - B_{j1}^k \quad (j \neq 1), \\ B_{ij}^k - B_{ji}^k &= 0 \quad (i, j \neq 1). \end{aligned} \tag{6}$$

Now B_{ij}^k satisfies $B_{ij}^k = -B_{ik}^j$ for any i, j, k since $\sum_{j,k} B_{ij}^k v^j \otimes v_k \in \mathfrak{g} \subset \mathfrak{so}(N)$. Then (7) gives $B_{ij}^k = 0$ for $i, j, k \neq 1$. Hence $B_{ij}^k = 0$ for $i \neq 1$ and $j, k = 1, \dots, N$ from (C2). Then

$$\begin{aligned} a &= \sum_{j,k} a_j^k v^j \otimes v_k \\ &= \sum_k a_1^k v^1 \otimes v_k + \sum_{j \neq 1} \sum_k (B_{1j}^k - B_{j1}^k) v^j \otimes v_k. \end{aligned}$$

Since $B_{j1}^k = 0$ for $j \neq 1$ and $k = 1, \dots, N$, we have

$$a = v^1 \otimes \sum_k a_1^k v_k + \sum_{j,k} B_{1j}^k v^j \otimes v_k - \sum_k B_{11}^k v^1 \otimes v_k.$$

Then we have finished the proof by putting $b = \sum_{j,k} B_{1j}^k v^j \otimes v_k$ and $w = \sum_k (a_1^k v_k - B_{11}^k) v_k$. \square

Proposition 4.3. *Let G be a Lie subgroup of $O(N)$ satisfying the condition (C1). Then the complex (5) is elliptic at $k = 1$ for any torsion-free G -structure Q .*

Proof. Let $a \in P_{\mathfrak{g}}^1$, $u \in V^* - \{0\}$ and $Sb_1(u)a = 0$. Since $Sb_1(u)a = 0$ means that $u \wedge a$ is an element of \mathfrak{g}^2 , we may write $a = b + u \wedge w$ for some $b \in \mathfrak{g}$ and $w \in V$ from Lemma 4.2. Hence we obtain

$$a = pr_{P_{\mathfrak{g}}^1}(a) = pr_{P_{\mathfrak{g}}^1}(b + u \wedge w) = pr_{P_{\mathfrak{g}}^1}(u \wedge w) = Sb_0(u)w.$$

\square

Proposition 4.4. *If a Lie group $G \subset O(N)$ is defined by*

$$G := \{g \in GL_N \mathbb{R}; \rho(g)\Phi^V = \Phi^V\}$$

for $\Phi^V \in \bigoplus_{i=1}^l \Lambda^{p_i}$, then G satisfies the condition (C1).

Proof. Let v_1, \dots, v_n be an orthonormal basis of $V = \mathbb{R}^n$, and v^1, \dots, v^n be its dual basis. We suppose that $A = \sum_{i,j} A_i^j v^i \otimes v_j$ is an element of \mathfrak{g} and $A_i^j = 0$ for $i, j \neq 1$. From the definition of G , we have

$$\sum_{1 \leq j \leq n} A_1^j v^1 \wedge \iota_{v_j} \Phi^V + \sum_{2 \leq i \leq n} A_i^1 v^i \wedge \iota_{v_1} \Phi^V = 0.$$

So we have

$$\begin{aligned} \sum_{1 \leq j \leq n} A_1^j v^1 \wedge \iota_{v_j} \Phi^V &= 0, \\ \sum_{2 \leq i \leq n} A_i^1 v^i \wedge \iota_{v_1} \Phi^V &= 0. \end{aligned}$$

Thus we have shown $\sum_{1 \leq j \leq n} A_1^j v^1 \otimes v_j$ and $\sum_{2 \leq i \leq n} A_i^1 v^i \otimes v_1$ are the elements of $\mathfrak{g} \subset \mathfrak{so}(N)$. Hence we obtain $A_1^j = A_i^1 = 0$ for any i, j . \square

It is easy to see $A_{\Phi_0}^{k+1}(d_k^\nabla \beta) = dA_{\Phi_0}^k(\beta)$ for $\beta \in \Omega^k(TM)$ by direct calculation using local orthonormal frame appear in the proof of Proposition 2.3, and we have the following proposition.

Proposition 4.5. *Let $\Phi_0 \in \widetilde{\mathcal{M}}_{\Phi^V}(M)$. Suppose that $\dim E_{\Phi^V}^k = \dim P_{\mathfrak{g}}^k$ for $k = l, l+1$. Then \bar{A}^l and \bar{A}^{l+1} are isomorphisms and $d \circ \bar{A}_{Q_{\Phi_0}}^l = \bar{A}_{\Phi_0}^{l+1} \circ d_{l+1}^{Q_{\Phi_0}}$.*

Now, we have $\dim E_{\Phi^V}^1 = \dim P_{\mathfrak{g}}^1$ by the definition of G . So we have the following.

Proposition 4.6. *Let $\Phi_0 \in \widetilde{\mathcal{M}}_{\Phi^V}(M)$ and suppose that $\dim E_{\Phi^V}^2 = \dim P_{\mathfrak{g}}^2$. Then the complex (1) is an elliptic complex at $k = 1$.*

Since $\dim E_{\Phi_{qK}}^2 = \dim P_{\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)}^2$ from the proof of Theorem 3.2, we have shown Lemma 3.3.

5 Bochner-Weitzenböck formulas on the quaternionic Kähler manifold

In this section we give a proof of Lemma 3.4.

Lemma 5.1. *Let $\Phi_0 \in \widetilde{\mathcal{M}}_{qK}$, and $\Delta_{\sharp} = dd_0^* + d_1^*d$ be as in Lemma 3.3. Then we have $\text{Ker } \Delta_{\sharp} = \mathbb{R}\Phi_0$ for $n \geq 3$.*

Proof. By the definition of $E_{\Phi_{qK}}^k$, we have $E_{\Phi_{qK}}^1 = A^1(\text{End}\mathbb{R}^{4n})$. Since there is a natural decomposition

$$\text{End}\mathbb{R}^{4n} = \mathfrak{so}(4n) \oplus \mathbb{R}(\text{Id}_{\mathbb{R}^{4n}}) \oplus \mathbf{symm}_0(4n),$$

where $\mathbf{symm}_0(4n) = \{A \in \text{End}\mathbb{R}^{4n}; {}^t A = A, \text{trace}(A) = 0\}$, we have

$$\begin{aligned} E_{\Phi_{qK}}^1 &= A^1(\mathfrak{so}(4n)) \oplus A^1(\mathbb{R}\text{Id}_{\mathbb{R}^{4n}}) \oplus A^1(\mathbf{symm}_0(4n)) \\ &= A^+ \oplus \mathbb{R}\Phi_{qK} \oplus A^-, \end{aligned}$$

by putting $A^+ = A^1(\mathfrak{so}(4n))$, $A^- = A^1(\mathbf{symm}_0(4n))$. Then the above decomposition induces $E_{\Phi_0}^1 = \hat{A}_{\Phi_0}^+ \oplus \hat{\mathbb{R}}_{\Phi_0} \oplus \hat{A}_{\Phi_0}^-$. Note that $A^+ \otimes \mathbb{C} \cong \lambda_0^2 \sigma^2$, $A^- \otimes \mathbb{C} \cong \lambda_1^2 \sigma^2 \oplus \lambda_0^2$ and $\mathbb{C}\Phi_{qK} \cong \sigma^0$.

Let the map $J : E_{\Phi_0}^1 \rightarrow E_{\Phi_0}^1$ be given by $J(\alpha) := *(\alpha \wedge \Phi_0^{n-2})$ where $*$ is the Hodge star operator with respect to g_{Φ_0} . Then we can calculate $J(\alpha)$ by using decomposition $E_{\Phi_0}^1 = \hat{A}_{\Phi_0}^+ \oplus \hat{\mathbb{R}}_{\Phi_0} \oplus \hat{A}_{\Phi_0}^-$.

(i) Let $\alpha = \Phi_0$. Then we have

$$J(\Phi_0) = *(\Phi_0 \wedge \Phi_0^{n-2}) = *\Phi_0^{n-1} = \frac{|\Phi_{qK}|^2}{c_n} \Phi_0$$

where c_n is given by $\Phi_{qK}^n = c_n \text{vol}_{g_0}$.

(ii) Let $\alpha = \sum_{i,j} a_i^j \xi^i \wedge \iota_{\xi_j} \Phi_0 \in \hat{A}_{\Phi_0}^+$, where ξ^i, ξ_j are as in the proof of Proposition 2.3, and $a_i^j = -a_j^i$. Then we have

$$\begin{aligned} J(\alpha) &= *(\sum_{i,j} a_i^j \xi^i \wedge \iota_{\xi_j} \Phi_0 \wedge \Phi_0^{n-2}) \\ &= \frac{1}{n-1} *(\sum_{i,j} a_i^j \xi^i \wedge \iota_{\xi_j} \Phi_0^{n-1}) \\ &= \frac{1}{n-1} \sum_{i,j} a_i^j \iota_{\xi_i} (\xi^j \wedge *\Phi_0^{n-1}) \\ &= -\frac{1}{n-1} \sum_{i,j} a_i^j \xi^j \wedge \iota_{\xi_i} *\Phi_0^{n-1} \\ &= \frac{1}{n-1} \frac{|\Phi_{qK}|^2}{c_n} \sum_{i,j} a_j^i \xi^j \wedge \iota_{\xi_i} \Phi_0 = \frac{1}{n-1} \frac{|\Phi_{qK}|^2}{c_n} \alpha. \end{aligned}$$

(iii) Let $\alpha \in \hat{A}_{\Phi_0}^-$. By calculating in the same way as (ii), we have

$$J(\alpha) = -\frac{1}{n-1} \frac{|\Phi_{qK}|^2}{c_n} \alpha.$$

From (i)(ii)(iii), it follows

$$J(\alpha) = \frac{|\Phi_{qK}|^2}{c_n} \left(\alpha_0 + \frac{1}{n-1} \alpha_+ - \frac{1}{n-1} \alpha_- \right)$$

for $\alpha = \alpha_0 + \alpha_+ + \alpha_- \in \Gamma(\hat{\mathbb{R}}_{\Phi_0} \oplus \hat{A}_{\Phi_0}^+ \oplus \hat{A}_{\Phi_0}^-)$.

Let α is an element of $\text{Ker } \Delta_{\sharp}$, which means $d\alpha = d_0^* \alpha = 0$. Then

$$d^* J(\alpha) = - * d(\alpha \wedge \Phi_0) = - * (d\alpha \wedge \Phi_0) = 0.$$

Let $p_0 : \Lambda^3 T^* M \rightarrow E_{\Phi_0}^0$ be the orthogonal projection. Since the formal adjoint d_0^* is given by $d_0^* = p_0 d^*$, we have two equations

$$\frac{c_n}{|\Phi_{qK}|^2} d^* J(\alpha) = d^* \left(\alpha_0 + \frac{1}{n-1} \alpha_+ - \frac{1}{n-1} \alpha_- \right) = 0, \quad (8)$$

$$d_0^* \alpha = p_0 d^* (\alpha_0 + \alpha_+ + \alpha_-) = 0. \quad (9)$$

Then by calculating $(n-1)p_0 \times (8) + (9)$, we obtain

$$np_0 d^* \alpha_0 + 2p_0 d^* \alpha_+ = 0. \quad (10)$$

Since $d^* \alpha_0 \in \Gamma(E_{\Phi_0}^0)$, the equation (10) is equivalent to

$$nd^* \alpha_0 + 2p_0 d^* \alpha_+ = 0. \quad (11)$$

There are non-trivial $Sp(n)Sp(1)$ -equivariant maps

$$\mathbf{T}_0 : \mathbb{R}\Phi_{qK} \rightarrow \Lambda^0, \quad \mathbf{T}_1 : E_{\Phi_{qK}}^0 \rightarrow \Lambda^1, \quad \mathbf{T}_2 : A^+ \rightarrow \Lambda^2.$$

Since \mathbf{T}_0 and \mathbf{T}_1 are isomorphisms and \mathbf{T}_2 is injective, each \mathbf{T}_i is determined uniquely up to scalar multiple by Schur's lemma. Then the bundle maps

$$\begin{aligned} \hat{\mathbf{T}}_{0,\Phi_0} &: \hat{\mathbb{R}}_{\Phi_0} \longrightarrow \Lambda^0 T^* M, \\ \hat{\mathbf{T}}_{1,\Phi_0} &: E_{\Phi_0}^0 \longrightarrow \Lambda^1 T^* M, \\ \hat{\mathbf{T}}_{2,\Phi_0} &: \hat{A}_{\Phi_0}^+ \longrightarrow \Lambda^2 T^* M \end{aligned}$$

are induced by $\widetilde{Q_{\Phi_0}}$ and each \mathbf{T}_i . From Schur's lemma, there are nonzero constants $C_0, C_+ \in \mathbb{R}$ which satisfy

$$\begin{aligned} \hat{\mathbf{T}}_{1,\Phi_0}(d^* \alpha_0) &= C_0 d \hat{\mathbf{T}}_{0,\Phi_0}(\alpha_0), \\ \hat{\mathbf{T}}_{1,\Phi_0}(p_0 d^* \alpha_+) &= C_+ d^* \hat{\mathbf{T}}_{2,\Phi_0}(\alpha_+), \end{aligned}$$

since $\mathbb{R}\Phi_{qK}$, $E_{\Phi_{qK}}^0$ and A^+ are the irreducible $Sp(n)Sp(1)$ -modules. So the equation (11) is equivalent to

$$nC_0 d\hat{\mathbf{T}}_{0,\Phi_0}(\alpha_0) + 2C_+ d^* \hat{\mathbf{T}}_{2,\Phi_0}(\alpha_+) = 0,$$

which gives $d\hat{\mathbf{T}}_{0,\Phi_0}(\alpha_0) = d^* \hat{\mathbf{T}}_{2,\Phi_0}(\alpha_+) = 0$. Since $\hat{\mathbf{T}}_{1,\Phi_0}$ is an isomorphism, we obtain

$$d^* \alpha_0 = p_0 d^* \alpha_+ = 0. \quad (12)$$

Next we consider the condition $d\alpha = 0$. There is the decomposition

$$E_{\Phi_0}^2 = (\lambda_0^3 \hat{\sigma}^3)_{\Phi_0} \oplus (\lambda_1^3 \hat{\sigma}^3)_{\Phi_0} \oplus (\lambda_0^3 \hat{\sigma}^1)_{\Phi_0} \oplus (\lambda_1^3 \hat{\sigma}^1)_{\Phi_0} \oplus (\lambda_0^1 \hat{\sigma}^3)_{\Phi_0} \oplus (\lambda_0^1 \hat{\sigma}^1)_{\Phi_0}$$

induced by the decomposition of $E_{\Phi_{qK}}^2$ as in Section 3. By taking the orthogonal projection $\Pi_{\lambda_0^3 \sigma^3} : E_{\Phi_0}^2 \rightarrow (\lambda_0^3 \hat{\sigma}^3)_{\Phi_0}$, we have

$$\Pi_{\lambda_0^3 \sigma^3}(d\alpha_+) = 0, \quad (13)$$

because the irreducible $Sp(n)Sp(1)$ -decomposition of $V^* \otimes (\mathbb{R}\Phi_{qK} \oplus A^-)$ does not contain the component of $\lambda_0^3 \sigma^3$.

Since the space A^+ is isomorphic to $\lambda_0^2 \sigma^2$ as an $Sp(n)Sp(1)$ -module, there are the irreducible decomposition

$$\begin{aligned} T^*M \otimes \hat{A}_{\Phi_0}^+ &= (\lambda_1^3 \hat{\sigma}^3)_{\Phi_0} \oplus (\lambda_0^3 \hat{\sigma}^3)_{\Phi_0} \oplus (\lambda_0^1 \hat{\sigma}^3)_{\Phi_0} \\ &\quad \oplus (\lambda_0^3 \hat{\sigma}^1)_{\Phi_0} \oplus (\lambda_1^3 \hat{\sigma}^1)_{\Phi_0} \oplus (\lambda_0^1 \hat{\sigma}^1)_{\Phi_0}, \end{aligned}$$

and orthogonal projections $pr_{\lambda_q^p \sigma^r} : T^*M \otimes \hat{A}_{\Phi_0}^+ \rightarrow (\lambda_q^p \hat{\sigma}^r)_{\Phi_0}$. Then differential operator $D_{a,b}$ on $\Gamma(\hat{A}_{\Phi_0}^+)$ are defined by

$$\begin{aligned} D_{1,1} &:= pr_{\lambda_1^3 \sigma^3} \circ \nabla, & D_{1,3} &:= pr_{\lambda_0^3 \sigma^3} \circ \nabla, & D_{1,-2} &:= pr_{\lambda_0^1 \sigma^3} \circ \nabla, \\ D_{-1,1} &:= pr_{\lambda_1^3 \sigma^1} \circ \nabla, & D_{-1,3} &:= pr_{\lambda_0^3 \sigma^1} \circ \nabla, & D_{-1,-2} &:= pr_{\lambda_0^1 \sigma^1} \circ \nabla, \end{aligned}$$

where ∇ is the Levi-Civita connection of g_{Φ_0} . According to [6], there are equations for $B_{a,b} := (D_{a,b})^* D_{a,b}$ and the scalar curvature $\kappa_{g_{\Phi_0}}$, where $(D_{a,b})^*$ is the formal adjoint of $D_{a,b}$,

$$\begin{aligned} \frac{1}{n+2} \kappa_{g_{\Phi_0}} &= -B_{1,1} + 2B_{1,3} + 2nB_{1,-2} \\ &\quad -B_{-1,1} + 2B_{-1,3} + 2nB_{-1,-2}, \end{aligned} \quad (14)$$

$$\begin{aligned} \frac{2}{n+2} \kappa_{g_{\Phi_0}} &= -2(B_{1,1} + B_{1,3} + B_{1,-2}) \\ &\quad + 4(B_{-1,1} + B_{-1,3} + B_{-1,-2}), \end{aligned} \quad (15)$$

$$\begin{aligned} \frac{8}{n+2} \kappa_{g_{\Phi_0}} &= -2(n+2)B_{1,1} + 4(n-1)B_{1,3} - 4n(n-1)B_{1,-2} \\ &\quad + 4(n+2)B_{-1,1} - 8(n-1)B_{-1,3} + 8n(n-1)B_{-1,-2}. \end{aligned} \quad (16)$$

Now we have $p_0 d^* \alpha_+ = 0$ and $\Pi_{\lambda_0^3 \sigma^3}(d\alpha_+) = 0$ from (12)(13), which gives $B_{-1,-2}\alpha_+ = 0$ and $B_{1,3}\alpha_+ = 0$, respectively. So by calculating $2(n^2 - n - 2) \times (14) - n(n+3) \times (15) + \frac{2n+1}{2} \times (16)$, we have

$$\begin{aligned} & -(2n+1)(n-2)B_{1,1}\alpha_+ - 2(n-2)(n+2)B_{-1,1}\alpha_+ \\ & -4(2n+1)(n+1)B_{-1,3}\alpha_+ \\ & = 0. \end{aligned}$$

Then by taking L^2 inner product with α_+ , it follows that

$$\begin{aligned} & -(2n+1)(n-2)\|D_{1,1}\alpha_+\|_{L^2}^2 - 2(n-2)(n+2)\|D_{-1,1}\alpha_+\|_{L^2}^2 \\ & -4(2n+1)(n+1)\|D_{-1,3}\alpha_+\|_{L^2}^2 \\ & = 0, \end{aligned}$$

which gives $B_{1,1}\alpha_+ = B_{-1,1}\alpha_+ = B_{-1,3}\alpha_+ = 0$ since $n \geq 3$. Hence (14) and (15) gives $2nB_{1,-2}\alpha_+ = \frac{1}{n+2}\kappa_{g_{\Phi_0}}\alpha_+$ and $-B_{1,-2}\alpha_+ = \frac{1}{n+2}\kappa_{g_{\Phi_0}}\alpha_+$, respectively. Then by vanishing $B_{1,-2}$ from above two equations, we obtain $\frac{2n+1}{n+2}\kappa_{g_{\Phi_0}}\alpha_+ = 0$. Since we suppose the scalar curvature is nonzero, we have $\alpha_+ = 0$.

From (11) and $\alpha_+ = 0$, we have $d^*\alpha_0 = 0$. If we write $\alpha_0 = f\Phi_0$ for $f \in C^\infty(M)$, then $d^*\alpha_0 = 0$ means $df = 0$ since the map $*(\cdot \wedge *\Phi_0) : T^*M \rightarrow \Lambda^3 T^*M$ is injective. So α_0 is given by $\alpha_0 = c\Phi_0$ for $c \in \mathbb{R}$.

Thus it follows $d\alpha_- = 0$, $d^*\alpha_- = 0$ from (8) and $d\alpha = d\alpha_0 = d\alpha_+ = 0$. But there is no nonzero harmonic forms on $\Gamma((\lambda_1^2 \sigma^2)_{\Phi_0} \oplus (\hat{\lambda}_0^2)_{\Phi_0})$ according to the vanishing theorems on the quaternionic Kähler manifold [6][14], hence we obtain $\alpha_- = 0$. \square

6 Quaternionic Kähler metrics and the reduced frame bundles

Let g be a Riemannian metric on M whose holonomy group $Hol(g)$ is isomorphic to $Sp(n)Sp(1)$. Then we set

$$\widetilde{\mathcal{M}}_{qK}(g) := \{\Phi \in \widetilde{\mathcal{M}}_{qK}; g_\Phi = g\}.$$

The purpose of this section is showing that there is a unique element in $\widetilde{\mathcal{M}}_{qK}(g)$.

We use following proposition, which can be seen in [8] p.46.

Proposition 6.1 ([8]). *Let g be a quaternionic Kähler metric on a connected manifold M of dimension $4n$. Then there is a one-to-one correspondence*

between $\widetilde{\mathcal{M}}_{qK}(g)$ and homogeneous space $Sp(n)Sp(1) \setminus N(Sp(n)Sp(1))$, where $N(Sp(n)Sp(1))$ is defined by

$$N(Sp(n)Sp(1)) := \{x \in O(4n); x\{Sp(n)Sp(1)\}x^{-1} \subset Sp(n)Sp(1)\}.$$

Proposition 6.2. *Let g be a quaternionic Kähler metric on a manifold M of dimension $4n \geq 12$. Then $\mathcal{M}_{qK}(g)$ has only one element.*

Proof. From Proposition 5.1, it suffices to show that

$$N(Sp(n)Sp(1)) = Sp(n)Sp(1).$$

Let ρ be as in Section 3. If we take $x \in N(Sp(n)Sp(1))$, then xhx^{-1} is an element of $Sp(n)Sp(1)$ for any $h \in Sp(n)Sp(1)$. So it follows that

$$\begin{aligned} \rho(xhx^{-1})\Phi_{qK} &= \Phi_{qK}, \\ \rho(h)\rho(x^{-1})\Phi_{qK} &= \rho(x^{-1})\Phi_{qK}, \end{aligned}$$

for any $h \in Sp(n)Sp(1)$. It means that $\rho(x^{-1})\Phi_{qK}$ is an element of

$$(\Lambda^4)^{Sp(n)Sp(1)} := \{\alpha \in \Lambda^4; \rho(h)\alpha = \alpha \text{ for any } h \in Sp(n)Sp(1)\}.$$

Then we may write $\rho(x^{-1})\Phi_{qK} = \lambda\Phi_{qK}$ for $\lambda \in \mathbb{R}$ since $(\Lambda^4)^{Sp(n)Sp(1)} \cong \mathbb{R}$ from the irreducible $Sp(n)Sp(1)$ -decomposition of Λ^4 in Section 3. Since we have

$$(\rho(x^{-1})\Phi_{qK})^n = \rho(x^{-1})\Phi_{qK}^n = \det(x)\Phi_{qK}^n,$$

it follows that $\lambda = \pm 1$. So we have

$$N(Sp(n)Sp(1)) = \{x \in O(4n); \rho(x^{-1})\Phi_{qK} = \pm\Phi_{qK}\},$$

and it follows that the irreducible $Sp(n)Sp(1)$ -module $\mathbb{R}\omega_I \oplus \mathbb{R}\omega_J \oplus \mathbb{R}\omega_K \subset \Lambda^2$ is an irreducible $N(Sp(n)Sp(1))$ -module. Then we may write

$$\begin{aligned} \rho(x^{-1})\omega_I &= A_1\omega_I + A_2\omega_J + A_3\omega_K, \\ \rho(x^{-1})\omega_J &= B_1\omega_I + B_2\omega_J + B_3\omega_K, \\ \rho(x^{-1})\omega_K &= C_1\omega_I + C_2\omega_J + C_3\omega_K, \end{aligned}$$

for some $A_i, B_i, C_i \in \mathbb{R}$. If $\rho(x^{-1})\Phi_{qK} = -\Phi_{qK}$, then $A_1^2 + A_2^2 + A_3^2 = -1$. Hence $\rho(x^{-1})\Phi_{qK}$ has to be $\rho(x^{-1})\Phi_{qK} = \Phi_{qK}$, which means $x \in Sp(n)Sp(1)$. Thus we have shown $N(Sp(n)Sp(1)) = Sp(n)Sp(1)$. \square

References

- [1] F. A. Bogomolov, Hamiltonian Kähler manifolds, *Soviet Math. Dokl.*, **19** (1978) pp. 1462-1465.
- [2] D. Ebin, The manifolds of Riemannian metrics, *Proceedings of the Symposium in Pure Mathematics of the American Mathematical Society*, **15**, pp. 11-40.
- [3] A. Fujiki, On the de Rham cohomology group of a compact Kähler symplectic manifold, *Advanced Studies in Pure Math.* **10**, 1987 Algebraic Geometry, Sendai, (1985) pp. 105-165.
- [4] W. Fulton and J. Harris, *Representation Theory A First Course*, Springer, Graduate Texts in Math. 129 (1991).
- [5] R. Goto, Moduli spaces of topological calibrations, Calabi-Yau, hyperKähler, G_2 and $Spin(7)$ structures, *International Journal of Mathematics* Vol. 15 No.3 (2004) pp. 211-257.
- [6] Y. Homma, Estimating the eigenvalues on Quaternionic Kähler Manifolds, *International Journal of Mathematics*, @Vol. 17 No.6 (2006) pp. 665-691.
- [7] R. Horan, A rigidity theorem for quaternionic Kähler manifolds, *Differential Geometry and its Applications*, Vol. 6 (1996) pp. 189-196
- [8] D. D. Joyce, *Compact manifolds with special holonomy*, Oxford Math. Monographs (Oxford Science Publication, 2000).
- [9] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vol I, II*, Wiley Interscience, 1996.
- [10] C. LeBrun, A Rigidity Theorem for Quaternionic-Kähler Manifolds, *Proc. Am. Math. Soc.* **103** (1988) pp. 1205-1208.
- [11] C. LeBrun and S. M. Salamon, Strong rigidity of positive quaternion-Kähler manifolds, *Invent. Math.*, **118** (1994) pp. 108-132.
- [12] Y. Nagatomo and T. Nitta, Vanishing theorems for quaternionic complexes, *Bull. London Math. Soc.*, **29** (1997) pp. 359-366.
- [13] R. S. Palais, On the Differentiability of Isometries, *Proceedings of the American Mathematical Society*, Vol. 8 (1957) pp. 805-807
- [14] U. Semmelmann and G. Weingart, Vanishing Theorems for Quaternionic Kähler Manifolds, *J. Reine Angew. Math.* **544** (2002), pp. 111-132.
- [15] S. M. Salamon, *Riemannian geometry and holonomy groups*, Pitman Research Notes in Mathematics Series 201.

- [16] S. M. Salamon, Quaternionic Kähler manifolds, *Invent. Math.* **67** (1982) pp. 143-171.
- [17] S. M. Salamon, Differential geometry of quaternionic manifolds, *Annales Scientifiques de l'École Normale Supérieure*, **19** (1986) pp. 31-55.
- [18] S. M. Salamon, Quaternion-Kähler geometry, In C. LeBrun and M. Wang, editors, *Essays on Einstein Manifolds*, Volume VI of Surveys in Differential Geometry, pp. 83-122.
- [19] A. Swann, Aspects symplectiques de la géométrie quaternionique, *C. R. Acad. Sci. Paris, t.* **308** (1989), Série I, pp. 225-228.
- [20] G. Tian, Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Peterson-Weil metric, in *Mathematical Aspects of String Theory*, ed. S.-T. Yau, Advanced Series in Mathematical Physics, Vol. 10 (World Scientific Publishing Co., Singapore, 1987), pp. 629-646.
- [21] A. N. Todolov, The Weil-Peterson geometry of the moduli space of $SU(n \geq 3)$ (Calabi-Yau) manifolds. I, *Comm. Math. Phys.* **126** (1989) pp. 325-346.